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Initial-Data Formulation of Tetrad Gravity Utilizing York's Extrinsic Time Approach

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An ability to analyze the geometrodynamic degrees of freedom and initial-data formulation is central to the canonical quantization of gravity. In the metric theory of gravity York provided the most powerful technique to analyze the dynamic degrees of freedom and to solve the initial-data problem. In this paper we extend York's analysis to tetrad gravity. Such an extension is necessary for the quantization of gravity when coupled to a half-integer-spin field. We present a comparative analysis of the geometric information carried by (1) a 3-metric of an initial hypersurface and (2) the spacelike triad of a time-gauged tetrad. We apply the tetrad initial-data formulation to Ashtekar's connection variables, and provide a comparison with other alternative choices of canonical tetrad variables.

1. INTRODUCTION

An ability to analyze the geometrodynamic degrees of freedom and the initial-data formulation in general relativity is a key prerequisite for canonical quantum gravity. Without such an analysis there is no way to avoid conceptual difficulties in formulating quantum geometrodynamics (Gerlach, 1969; Kheyfets and Miller, 1995).

The most powerful approach to the initial-data formulation in metric gravity was put forward by J. York in the early seventies (York, 1971, 1972a,b, 1973). The strength of York's procedure is based on the theory of conformally invariant orthogonal decomposition of symmetric tensors on Riemannian 3-manifolds. The significance of this decomposition theory for symmetric tensors is similar to the significance of the Hodge theory for differential forms.

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In recent years more attention has been focused on tetrad-based gravity quantization rather than on metric-based quantizations. This research direction includes Ashtekar's connection representation of general relativity, which seems to have made some advances toward developing the quantum theory of gravity (Ashtekar, 1994). A tetrad formulation of the initial value problem can also be useful and necessary when the gravity field source is a half-integer-spin matter field. It is for these reasons we have undertaken our investigation aimed at extending York's solution of the initial-data problem to the tetrad formalism.

Our application of York's metric procedure to the tetrad formalism is achieved via (1) a careful inspection of the spacelike triad as a carrier of geometric information, and (2) a comparison of the ways in which geometric information is stored in the triad *vis à vis* the 3-metric. We describe the results of this inspection in Section 3, emphasizing that the triad and the 3-metric carry geometric information in similar fashions. In particular, we describe a procedure of separating the scale factor from the conformal 3-geometry in the triad description. We find that rescaling the triad is equivalent to rescaling the 3-metric induced by the triad.

An inspection of the Gauss (gauge) constraint of the tetrad formalism shows that this constraint admits an explicit solution. This solution allows one to reduce the rest of the tetrad-based initial-data problem (i.e., solving the vector and Hamiltonian constraints) to solving the initial-data problem of the metric formulation (the 3-metric being induced by the triad).

Our final conclusion is that York's extrinsic time approach can ordinarily be extended to the tetrad formalism, and to Ashtekar's variables in particular. Furthermore, such an extension involves a minimal change in the manipulations as compared to the standard metric formalism.

In Section 2 we provide a brief review of York's procedure in the metric formulation. Section 3 analyzes the triad as a carrier of geometric information and compares it with the 3-metric. Section 4 extends York's approach to time-gauged tetrads. Section 5 discusses several alternative choices of canonical variables in tetrad gravity including Ashtekar's variables. Finally, Section 6 contains a brief discussion of some alternative approaches to the initial-data problem.

2. YORK'S ANALYSIS OF THE GRAVITATIONAL DEGREES OF FREEDOM AND INITIAL CONDITIONS IN THE METRIC REPRESENTATION: A REVIEW

It is not our intention to provide here a complete description of York's solution of the initial-data problem, as it can be found elsewhere (York, 1971, 1972a,b, 1973; O'Murchadha and York, 1973; Misner *et al.*, 1973; Wheeler, 1988). We wish only to highlight those points of York's analysis that have direct parallels with our presentation of the initial-data problem of tetrad gravity.

One of the ways to split the dynamics of the gravitational field into its evolution and initial-data formulation is to use the ADM procedure (Misner *et al.*, 1973; Arnowitt *et al.*, 1962). This facilitates the transition from the Lagrangian formulation of gravitational dynamics to the constrained Hamiltonian description. The procedure involves a slicing of spacetime by spacelike 3-dimensional hypersurfaces of simultaneity, and identifies (1) the components of the 3-metric as the dynamic variables and (2) the momenta conjugate to the 3-metric. Evolution of the system (when passing from slice to slice) is determined by the Hamilton equations. In addition, four constraint equations are imposed on the metric and momentum within each hypersurface. It is an important consequence of the formalism that, once the constraints are satisfied on one slice, the evolution equations guarantee that they will be satisfied on any subsequent slice. Thus, it is sufficient to satisfy the constraint equations only on an initial hypersurface.

The four constraint equations are (Misner *et al.*, 1973)

$$\pi^{ab}{}_{|b} = \begin{cases} 0 \\ \quad \text{when there is no flow of energy in space} \\ 8\pi(\text{density of flow of energy})^a \\ \quad \text{otherwise} \end{cases} \tag{2.1}$$

and

$$\begin{aligned} \mathcal{H}(\pi^{ab}, g_{ab}) &= g^{-1/2}\left(\mathrm{Tr}\,\Pi^2 - \frac{1}{2}(\mathrm{Tr}\,\Pi)^2\right) - g^{1/2}R \\ &= 16\pi(\text{density of energy}) \end{aligned} \tag{2.2}$$

where

$$\pi^{ab} = \begin{pmatrix} \text{“geometrodynamic} \\ \text{field momentum”} \\ \text{conjugate to } g_{ab} \end{pmatrix} = g^{1/2}(g^{ab}\,\mathrm{Tr}\,K - K^{ab}) \tag{2.3}$$

g_{ab} is the 3-metric of the initial hypersurface, g is the determinant of g_{ab}, R is the curvature invariant of the 3-metric g_{ab}, K^{ab} is the extrinsic curvature of the initial hypersurface as embedded in the ambient spacetime, Π is just another notation for π^{ab}, $\Pi^2 = \pi^{ac}\pi_c{}^b$, and $K = \mathrm{Tr}\,K$ is the trace of K^{ab}.

Constraint (2.1) (three equations altogether) is called the momentum constraint, the vector constraint, or the diffeomorphism constraint, depending on the context in which it is considered. Constraint (2.2) (one additional equation) is commonly known as the Hamiltonian constraint. The two constraints do not allow one to use arbitrary values for g_{ab}, π^{ab} as the initial data in geometrodynamics. Only a part of the components of g_{ab} and π^{ab} can be given freely. One function of these variables is singled out in this process to be used as the time variable for the dynamics. Such a function facilitates the slicing of spacetime by a one-parameter family of spacelike 3-hypersurfaces. It is associated with fixing of one 4-dimensional diffeomorphism degree of freedom or, equivalently, with imposing a slicing condition. Such an assignment of a time variable, a choice of the independent (true dynamic) variables, and the subsequent solving of the four constraint equations constitutes what ordinarily is identified as the initial-data problem.

The analysis of the initial-data problem prior to the early 1970s suffered from an insufficient geometric understanding of the problem and could handle only some particular cases. The role of a slicing condition was poorly understood. A slicing condition was usually imposed implicitly via introducing additional symmetries or other conditions not related to the structure of the constraint equations. It was only in 1971–1973, after York introduced a complete geometric analysis of the information contained in the metric tensor (York, 1971, 1972a,b, 1973; O'Murchadha and York, 1973), that it became possible to put forward a solution of the initial-data problem that could be applied in generality. In what follows, when discussing the initial-data problem, we will have in mind W-model universes, i.e., spacetimes admitting a unique slicing by spacelike, spatially closed hypersurfaces of Tr K = const within each of the hypersurfaces. For such spacetimes the value of Tr K, or rather

$$\tau = \tfrac{2}{3} g^{-1/2}\,\mathrm{Tr}\,\Pi = \tfrac{4}{3}\mathrm{Tr}\,K \tag{2.4}$$

identifies the slicing hypersurfaces uniquely and can be used as the time parameter (York's extrinsic time).

The six components g_{ab} of the 3-metric tensor on the initial hypersurface carry information concerning the internal geometry of the spacelike hypersurface as well as information concerning the choice of three coordinates x^a. Coordinatization of a 3-slice takes three pieces of information. This leaves $6 - 3 = 3$ pieces of information to describe the 3-geometry. One can think, in principle, of a possibility of fixing coordinates in such a way that the metric g_{ab} becomes diagonal. This picture makes it clear that in three dimensions the geometry is described by three scale factors (in three orthogonal directions). These three factors can be thought of as (1) a common scaling (one parameter per point) combined with (2) the conformal part of geometry (two parameters per point). Technically, the scale parameter of the 3-metric is split from the rest of information via a special factor ψ^4:

$$g_{ab} = \psi^4 (g_{ab})_{\text{base}};\ \det\{(g_{ab})_{\text{base}}\} = 1 \tag{2.5}$$

The key to York's success in solving the initial-data problem was what he called (York, 1973) a "conformally invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds" into "transverse traceless," "conformal Killing," and "trace" parts, measuring the deformation of the conformal part of geometry, recoordinatization, and change of scale, respectively.

For π^{ab} the transverse traceless part is expressed by

$$\tilde{\pi}^{ab} = g^{1/3}(\pi^{ab} - \tfrac{1}{3} g^{ab} \mathrm{Tr}\, \Pi) \tag{2.6}$$

called by York the "momentum density of weight 5/3," and satisfies the equations

$$\tilde{\pi}^{a}{}_{a} = 0 \qquad \text{(traceless)} \tag{2.7}$$

$$\tilde{\pi}^{ab}{}_{|b} = \begin{cases} 0 \text{ (transverse)} \\ \quad \text{when there is no flow of energy in space} \\ 8\pi (\text{density of flow of energy})^a \\ \quad \text{otherwise} \end{cases}$$

The momentum density $\tilde{\pi}^{ab}$ depends only on the conformal equivalence class of the metric, i.e., on

$$\tilde{g}_{ab} = g^{-1/3} g_{ab} \tag{2.8}$$

and the conditions of transversality (2.7), although they apparently contain the covariant derivatives determined by the Levi-Civita connection of the 3-metric, in fact depend only on the conformal equivalence class of the metric, and not on the choice of ψ^4.

To set up the initial-data problem on the hypersurface determined by a fixed value of τ, one fixes a coordinatization on the hypersurface, which determines three pieces of information in g_{ab}, specifies freely two more pieces of information [conformal 3-geometry, or conformal part of $(g_{ab})_{\text{base}}$] in (2.5), imposes an additional condition $g_{\text{base}} = 1$, specifies freely two pieces of information out of five in $\tilde{\pi}^{ab}$, and solves the transversality equations for the three remaining pieces of information. The remaining two pieces of information are the trace part of π^{ab} (determined by the choice of τ) and the scale factor ψ, which is determined by the equation derived from the Hamiltonian constraint

$$8\nabla^2\psi - R\psi + M\psi^{-7} + Q\psi^{-3} - \tfrac{3}{8}\tau^2\psi^5 = 0 \tag{2.9}$$

where ∇^2 stands for the Laplacian

$$\nabla^2\psi \equiv g^{-1/2}\left(\frac{\partial}{\partial x^a}\right)g^{1/2}g^{ab}\left(\frac{\partial}{\partial x^b}\right) \tag{2.10}$$

and

$$M \equiv g^{-5/3}g_{ab}g_{cd}\tilde{\pi}^{ac}\tilde{\pi}^{bd} \tag{2.11}$$

$$Q \equiv 16\pi\rho_{\text{base}} = 16\pi\psi^8\rho \tag{2.12}$$

Here ∇^2, R, M, and Q all refer to the basic geometry. As is well known, a solution of (2.9) exists, and it is unique (York, 1971) under quite generic conditions.

3. TETRAD DESCRIPTION: ADDITIONAL GAUGE DEGREES OF FREEDOM BUT THE SAME GEOMETRY

Various tetrad (or vierbein) formulations of gravitation theory provide considerable advantages in solving some particular problems and are necessary for some basic procedures, such as the direct coupling of half-integer-spin systems to gravity (Deser and van Niewenhuizen, 1974). The tetrad description of gravity appears to be of key importance in Ashtekar's representation of general relativity, which has drawn a great deal of attention in recent literature. We will apply our analysis of the initial-data problem in the tetrad formalism to this particular case as an example.

In the tetrad formalism, the metric tensor as a carrier of geometric information is replaced by four linearly independent covariant vector fields ${}^4e^I_\mu$ or contravariant vierbein fields ${}^4e^\mu_I$ (Greek indices μ, ν are used as spacetime indices, while capital Latin indices I, K are tetrad, or internal, indices). The vierbein at a point of spacetime can be interpreted as a 1-to-1 soldering map between the tangent space of spacetime at this point and the internal space. Equivalently, the vierbein describes an orthonormal frame at each point of spacetime. The vierbeins are related to the metric of spacetime $g_{\mu\nu}$ and Minkowski metric η_{IK} of the internal space via

$$\begin{aligned} {}^4e^\mu_I\, g_{\mu\nu}\, {}^4e^\nu_K &= \eta_{IK} \\ {}^4e^\mu_I \eta^{IK}\, {}^4e^\nu_K &= g^{\mu\nu} \end{aligned} \tag{3.1}$$

The vierbein matrix ${}^4e^\mu_I$ is not symmetric and has 16 components—6 components more compared to the 10 components of the metric $g_{\mu\nu}$. The larger number of degrees of freedom in the vierbein description of the gravitational field is caused by the simultaneous use of two frames. In addition to the coordinate frame $\{\partial_\mu\}_{\mu=0,\dots,3}$ used in both the metric and the tetrad formalism, an additional orthonormal tetrad frame is introduced specifically in the vierbein formalism, and the extra 6 degrees of freedom are responsible for the freedom of choice of the tetrad frame (via rotations), or gauge.

In the transition to the 3 + 1 formalism in a way similar to the ADM procedure, spacetime is sliced by 3-dimensional spacelike hypersurfaces. Often the coordinates are chosen in such a way that x^0 is transversal to these hypersurfaces, while three coordinates x^1, x^2, x^3 also coordinatize the hypersurfaces. In this case the coordinate frame vectors $\partial_a = \partial/\partial x^a$ are tangent to the slices and the g_{ab} part of the metric $g_{\mu\nu}$ induces a metric of Euclidean signature on slices (including the initial slice). In addition, the gauge is fixed partially so that ${}^4e_i^0 = 0$ (or ${}^4e_{\hat{0}}^a = 0$), which means that the vierbein is picked in such a way that the triad of the spacelike vectors of the vierbein is tangent to the slices. Such a partial fixing of the local Lorentz gauge is often referred to as the time gauge (Henneaux *et al.*, 1989). The transition between the coordinate basis on the hypersurface and the triad basis is given by a spatial 3×3 submatrix e_a^i (and the inverse to it e_i^a) of ${}^4e_\mu^I$ and

$$g_{ab} = \delta_{ik} e_a^i e_b^k \tag{3.2}$$

where indices like a, b, c are coordinate indices on slices, while indices like i, j, k are triad indices (numbers with carets over them are used for numerical values of triad indices).

The nine components of e_a^i contain information determined by the choice of the coordinatization of the slice (three pieces of information), the choice of gauge, or the vierbein triad (three more pieces of information), and the truly geometric part of the information (9 − 3 − 3 = 3 remaining pieces of information). The geometric part of the information is exactly the same as that carried by the metric. An easy way to see this is to fix coordinates in such a way that the coordinate basis becomes orthogonal, and then to fix the gauge (turn the vierbein triad) so that the vierbein triad vectors become parallel to the coordinate basis vectors. Such a procedure diagonalizes the vierbein triad matrix and the 3-metric tensor simultaneously. Keeping in mind that the vierbein vectors are unit vectors [formally from (3.2)] and after fixing coordinates and gauge, we conclude that

$$(e_1^{\hat{1}})^2 = g_{11}; \qquad (e_2^{\hat{2}})^2 = g_{22}; \qquad (e_3^{\hat{3}})^2 = g_{33} \tag{3.3}$$

This means that the geometry described by the diagonalized vierbein triad matrix is contained in three scale factors per point, one for each of the orthogonal directions, and, by the same logic as in the previous section, is nothing but a combination of simple scaling and conformal geometry.

All of the formal manipulation machinery in the tetrad representation can be developed in parallel with that of the metric representation, except there is really no need to do this. All the necessary relations can be transferred to the tetrad formalism from the metric formalism by partial change of the basis from the coordinate frame to the triad frame ("partial" means that it is applied not to all indices, but only to the appropriate ones, the last being very easy to identify in any practical situation) and keeping in mind that the triad spin connection is nothing but the Levi-Civita connection of the metric (3.2) when all the indices are transformed to the coordinate frame.

In some applications the orthonormal vierbein triad e^i_a is replaced with a triad that is not orthonormal but conformally equivalent to e^i_a (i.e., the triad e^i_a multiplied by a scale factor depending, in general, on coordinates). The modified triad is isotropic, i.e., all three vectors at a point are of equal length, the last being given as a function of metric. It is easy to see that for such triads the analysis of information contained in the triad remains exactly the same as for orthonormal triads, with a trivial change in equations (3.2) and (3.3).

For example, in Ashtekar's formalism (Rovelli, 1991), the densitized triad $\tilde{E}^a_i$ related to the orthonormal triad e^a_i as

$$\tilde{E}^a_i = \sqrt{g}\, e^a_i \tag{3.4}$$

plays a key role (here g is the determinant of g_{ab}). The spin connection of the densitized triad is the same as the spin connection of the original triad. The nine degrees of freedom represented by the matrix of the densitized triad consist of the three coordinatization degrees of freedom, three gauge degrees of freedom (rotation of the densitized triad), and the three true geometric degrees of freedom (scale and conformal geometry). However, formulas (3.2) and (3.3) will be replaced with

$$g g_{ab} = \delta_{ik} \tilde{E}^i_a \tilde{E}^k_b \tag{3.5}$$

and

$$(\tilde{E}^1_1)^2 = g g_{11}, \qquad (\tilde{E}^2_2)^2 = g g_{22}, \qquad (\tilde{E}^3_3)^2 = g g_{33} \tag{3.6}$$

The way the geometric information is stored in the triad matrix is closely related to that in the metric. In particular, a rescaling of the geometry by the factor ψ^4, as in equation (2.4), can be equivalently expressed in terms of the orthonormal triad matrix as

$$e^i_a = \psi^2 (e^i_a)_{\text{base}} \tag{3.7}$$
$$e^a_i = \psi^{-2} (e^a_i)_{\text{base}}$$

or in terms of the densitized triad matrix as

$$\tilde{E}^i_a = \psi^8 (\tilde{E}^i_a)_{\text{base}} \tag{3.8}$$
$$\tilde{E}^a_i = \psi^4 (\tilde{E}^a_i)_{\text{base}}$$

In what follows, an important role will be played by the two-frame tensors (with one coordinate and one triad index) obtained from symmetric coordinate tensors via a transformation of one of the indices, such as, for instance,

$$K^i_{\ a} = K_{ba}e^{ib} \qquad (K_{ba} = K_{ab}) \tag{3.9}$$

or their densitized versions. York's results concerning the conformally invariant decomposition of symmetric tensors can be used to analyze the structure of such tensors and, ultimately, the initial-data problem in the tetrad formulation and, in particular, Ashtekar's variables.

4. YORK'S PROCEDURE IN TETRAD GRAVITY

It has been shown in the previous section that the geometry of the initial-data problem remains essentially the same as one passes from the metric formulation of dynamics to its tetrad description. One of the consequences of this fact is that York's analysis of the gravitational degrees of freedom and the initial-data problem can be transferred in a straightforward way to the tetrad formalism. Retaining the full local Lorentz gauge freedom available in tetrad gravity is not essential in a discussion of the York procedure. Without a loss of generality we can restrict our consideration to time-gauged tetrads. In other words, we fix three out of six gauge parameters (Lorentz boost parameters) and keep three remaining gauge parameters (spatial rotations of the spacelike triad of a time-gauged tetrad). Canonical variables of such time-gauged gravity are the spatial triad components e_{ia} (i being a triad index and a being a spatial coordinate index) and their conjugate momenta π^{ia}.

The full system of constraints will include three vector constraints (2.1) and the Hamiltonian constraint (2.2) with the 3-metric g_{ab} and the metric momentum π^{ab} expressed in terms of the canonical variables of time-gauged tetrad gravity as follows:

$$g_{ab} = e_{ia}e^i_b \tag{4.1}$$

and

$$\pi^{ab} = \frac{1}{4}\,(e^a_i\pi^{ib} + e^b_i\pi^{ia}) \tag{4.2}$$

In addition to the constraints (2.1), (2.2) there will be three additional gauge constraints,

$$J^{ik} = \pi^{ia}e^k_{\ a} - \pi^{ka}e^i_{\ a} = 0 \tag{4.3}$$

generated by the three remaining gauge degrees of freedom. Constraint (4.3) has zero right-hand side, which means that we implicitly excluded from our consideration half-integer-spin matter fields. We are going to stick with this assumption for the remainder of this paper.

It is easy to see that York's procedure can be transferred to the time gauged tetrad formulation in a rather straightforward fashion. The reasons for this are as follows:

1. The vector constraints (2.1) and the Hamiltonian constraint (2.2) are the same as in the metric description, except that the 3-metric is determined by the triad via (3.2), and the covariant derivative is determined by the Levi-Civita connection of this metric.

2. The gauge constraint (4.3) (which is a new element compared to the metric representation) can be solved explicitly by demanding that π^{ia} be represented as

$$\pi^{ia} = e^{i}{}_{b}\pi^{ba} \tag{4.4}$$

where π^{ba} is a symmetric matrix that can be analyzed using all the standard York technique (a "conformally invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds" into transverse traceless, longitudinal, and "trace" parts).

3. The geometric information carried by the triad e_i^a is the same as that of the 3-metric, and is expressed as a combination of the conformal part and a common scale factor at each point. A rescaling of the triad matrix via

$$e^{ia} = \psi^2 (e^{ia})_{\text{base}} \tag{4.5}$$

leads to the rescaling of the induced metric

$$g_{ab} = \psi^4 (g_{ab})_{\text{base}} \tag{4.6}$$

just as in the standard York analysis of the metric representation. The base triad matrix ($3 \times 3 = 9$ elements) is fixed via the choice of coordinatization on the initial hypersurface (three pieces of information), choice of the triad orientation, or gauge (three more pieces of information), freely specifying two more pieces of information, and imposing the standard condition

$$e_{\text{base}} = \det | e_{ia} |_{\text{base}} = 1 \tag{4.7}$$

which, of course, implies

$$g_{\text{base}} = \det | g_{ab} |_{\text{base}} = e_{\text{base}}^2 = 1 \tag{4.8}$$

as is required in the standard York metric analysis.

The York solution of the initial-data problem in Ashtekar's variables proceeds as follows: first, we slice spacetime by the spacelike hypersurfaces of the constant extrinsic curvature and parametrize the slices by York's extrinsic time τ as in equation (2.4); then, on a given initial slice (determined by the value of τ), we fix the base triad $(e_{ia})_{\text{base}}$ as described above. Let us recall that the procedure of fixing the base triad includes specifying freely two pieces of information in it.

The next step is to specify the initial values for π^{ia}. After fixing the base triad and choosing π^{ia} according to (4.4) which satisfies automatically the gauge constraints (4.3), the remaining six pieces of information carried by the momenta are contained in π^{ab}. Using the York slicing condition and time parametrization according to

$$\tau = \tfrac{4}{3}\text{Tr } K = \tfrac{2}{3}g^{-1/2}\text{Tr } \Pi = \tfrac{2}{3}\text{Tr } \Pi_{\text{base}} \tag{4.9}$$

we specify freely two pieces of information in the transverse traceless part $\tilde{\pi}^{ab}$ [cf. equation (2.6)] of π^{ab}, and solve three vector constraints with respect to the remaining three pieces of information contained in π^{ab} in the standard York metric analysis. At this stage π^{ab} and $(e_{ia})_{\text{base}}$ are completely determined. The last step involves using these parameters to form the standard York metric equation (2.9) for the scale factor and solving it, followed by recovering the final expression for the spacelike triad of the time-gauged tetrad according to (4.5) and its conjugate momentum according to (4.4).

5. ALTERNATIVE CHOICES OF CANONICAL VARIABLES

The choice of canonical variables in the time-gauged tetrad gravity used in the previous section is not the only one considered in the literature. Two examples of other choices are the variables $(\tilde{E}^{ia}, 2K_{ia})$ used by Henneaux *et al.* (1989) in transition from variables of the previous section to Ashtekar variables, and the Ashtekar variables themselves. The variables $(\tilde{E}^{ia}, 2K_{ia})$ are the densitized spacelike triad

$$\tilde{E}^{ia} = \sqrt{g}\, g^{ab} e^{i}{}_{b} \tag{5.1}$$

and

$$K_{ia} = e_i^{\,b} K_{ba} + \tfrac{1}{4} g^{-1/2} J_{ik} e^{k}{}_{a} \tag{5.2}$$

where K_{ba} is the extrinsic curvature of the initial hypersurface. The new canonical variable $2K_{ia}$ is related to the momentum π^{ia} of the previous section by

$$2K_{ia} = \tfrac{1}{2} g^{-1/2} e_i^{\,b} \pi^{jc} (e_{jc} g_{ba} - e_{jb} g_{ac} - e_{ja} g_{bc}) + \tfrac{1}{2} g^{-1/2} J_{ik} e^{k}_{a} \tag{5.3}$$

It is obvious that the procedure of the previous section will work virtually without changes for this new set of canonical variables, except that the rescaling relation (4.5) should be replaced by

$$\tilde{E}^{ia} = \psi^4 (\tilde{E}^{ia})_{\text{base}} \tag{5.4}$$

It is easy to see that in this setting the procedure remains practically unchanged for the Ashtekar variables themselves. The Ashtekar variables are the $\tilde{E}^{ia}$ and the complex Ashtekar connection

$$A_{ia} = 2K_{ia} + \frac{i}{2}\,\epsilon_{ijk}\omega_a^{jk} \tag{5.5}$$

with

$$\omega_a^{ik} = \frac{1}{2}\,[g_{ab,c}e^{ib}e^{kc} + e^{ib}e^k{}_{b,a} - g_{ab,c}e^{kb}e^{ic} - e^{kb}e^i{}_{b,a}] \tag{5.6}$$

The curvature of the Ashtekar connection can be expressed as (Henneaux *et al.*, 1989)

$$F_{iab} = \frac{i}{4}\,\epsilon^{cdf}({}^3R_{cdab} + 2\overline{K}_{ca}\overline{K}_{db})e_{if} + \frac{1}{2}\,(\overline{K}_{cb\,|\,a} - \overline{K}_{ca\,|\,b})e_i{}^c \tag{5.7}$$

with

$$\overline{K}_{ab} = K_{ib}e^i{}_a \tag{5.8}$$

These relations can be used to prove (Henneaux *et al.*, 1989) that on the constraint surface $J_{ik} = 0$ the vector and Hamiltonian constraints derived from the variational principle in Ashtekar variables are equivalent to the constraints (2.1), (2.2) of tetrad gravity. This equivalence implies that the procedure described in the previous section can be simply transferred to Ashtekar's variables. With regard to the gauge constraint in the Ashtekar representation, the real part of it is satisfied trivially during the reduction procedure given by (4.4) and the imaginary part is a requirement that ω_a^{jk} is the Levi-Civita connection of the metric induced by the triad $e^i{}_a$. In other words, York's procedure extends trivially and unchanged to Ashtekar's variables. It is interesting that in the time-gauged tetrad setting York's procedure does not require any special effort to take care of the reality conditions. They are satisfied automatically if both the time gauge and conditions (4.4) are applied.

6. DISCUSSION

Our description of York's procedure for the initial-data problem in tetrad gravity shows that the procedure works for all kinds of canonical variables, including the Ashtekar connection variables. It works in tetrad gravity in about the same way as in the standard metric representation. This means that the York procedure can be used in tetrad gravity, which can be very efficient in investigating general theoretical issues, such as the existence and uniqueness of the solution to the initial-value problem. However, in practice, in classical relativity this procedure does not make using tetrad variables more attractive than metric variables, as most of involved operations duplicate those of metric gravity.

It might seem that there should be a way to take advantage of the special features of tetrad variables, especially in case of Ashtekar variables, where the constraints of the theory appear to be simple. The vector and Hamiltonian constraints are polynomial with respect to the densitized triad. The hope has been expressed that these special features might provide an opportunity to find a different way of solving the initial-value problem, which can offer advantages over York's approach, at least at some particular conditions. We will discuss only one such approach. The most comprehensive description of it was given by Capovilla *et al.* (1993). Their prescription goes, in a sense, in an opposite direction compared to that of York's procedure. Instead of solving the gauge constraints first, one starts from the vector and Hamiltonian constraints and, using the expansion of the densitized triad in a basis determined by the curvature of the Ashtekar connection, solves algebraically both these constraints. After that, the gauge constraints are imposed on the parameters of this algebraic solution and the problem is reduced to a system of nonlinear partial derivative equations of the first order. We do not reproduce here a detailed description of this procedure. It can be found in the paper referred to above. We wish only to note here that the conditions (4.4) are not satisfied in this approach, which means that, in addition to the Gauss, momentum, and Hamiltonian constraints, one has to impose the reality condition.

The new procedure is interesting, but it has not been worked out to the same degree as York's approach. In particular, existence and uniqueness of the solution to the gauge constraints has not been investigated. The usual practice is to avoid an investigation of the general case and to impose instead additional restrictions, typically some assumptions of a symmetry in the hope of achieving better understanding of the difficulties. The slicing condition never has been discussed explicitly in connection with this new procedure. However, realizing the importance of this condition was crucial for the whole York undertaking. One might very well believe that the key to the eventual success of the alternative approach in Ashtekar variables of tetrad gravity could lie in finding an appropriate slicing condition and locating a natural class of spacetimes in which this approach could provide considerable advantages over York's procedure. This slicing condition cannot be found by imposing additional symmetries. Rather, as in the case of York's approach, it should be determined by the structure of the constraint equations. In other words, the whole program analogous to that associated with the York procedure in metric gravity still awaits to be developed in Ashtekar connection variables.

Concerning the issue of gravity quantization, at least if it is understood as the canonical quantization of classical general relativity, a key ingredient of any successful procedure of quantization is the ability to single out the true dynamical degrees of freedom (Kheyfets and Miller, 1995). It is clear from Sections 3 and 4 that such a goal can be achieved in the time-gauged tetrad variables as successfully as in the standard 3 + 1 split of the metric variables via employing a slightly modified York procedure. It is also clear that, generally speaking, using the tetrad variables does not lead to any essential advantages except in situations where the nature of the matter sources of the gravity field makes using tetrad variables mandatory or natural (as in the case of the half-integer-spin matter fields). Alternative choices of tetrad, or densitized tetrad, variables do not seem to give essential advantages in this direction. In particular, the connection variables seem to be ideally suited for handling the momentum constraints (diffeomorphic invariance), but are ill-suited for handling the scale factor.

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